Hypomorphy of graphs up to complementation

Jamel Dammak

Département de Mathématiques, Faculté des Sciences de Sfax
B.P. 802, 3018 Sfax, Tunisie
E-mail : jdammak@yahoo.fr

Gérard Lopez

Institut de Mathématiques de Luminy, CNRS-UPR 9016,
163 avenue de Luminy, case 907, 13288 Marseille cedex 9, France
E-mail : gerard.lopez1@free.fr

Maurice Pouzet and Hamza Si Kaddour

PCS, Université Claude Bernard Lyon1,
Domaine de Gerland - bât. Recherche B
50 avenue Tony-Garnier
F 69366 - Lyon cedex 07, France
E-mail : Maurice.Pouzet@univ-lyon1.fr , Hamza.Sikaddour@univ-lyon1.fr

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Abstract

Let V be a set of cardinality v (possibly infinite). Two graphs G and G' with vertex set V are *isomorphic up to complementation* if G' is isomorphic to G or to the complement $\overline{G}$ of G. Let k be a non-negative integer, G and G' are *k-hypomorphic up to complementation* if for every k-element subset K of V, the induced subgraphs $G_{\restriction K}$ and $G'_{\restriction K}$ are isomorphic up to complementation. A graph G is *k-reconstructible up to complementation* if every graph G' which is k-hypomorphic to G up to complementation is in fact isomorphic to G up to complementation. We give a partial characterisation of the set $\mathcal{S}$ of pairs (n, k) such that two graphs G and G' on the same set of n vertices are equal up to complementation whenever they are k-hypomorphic up to complementation. We prove in particular that $\mathcal{S}$ contains all pairs (n, k) such that $4 \leq k \leq n - 4$. We also prove that 4 is the least integer k such that every graph G having a large number n of vertices is k-reconstructible up to complementation; this answers a question raised by P. Ille [8].

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1 Introduction

Ulam Reconstruction Conjecture [14] (see [2, 3]) asserts that two graphs G and G' on the same finite set V of v vertices, $v \geq 3$, are isomorphic provided that the restrictions $G_{\restriction K}$ and $G'_{\restriction K}$ of G and G' to the $(v-1)$-element subsets of V are isomorphic. If this latter condition holds for the k-element subsets of V for some k, $2 \leq k \leq v-2$, then, as it has been noticed several times, G and G' are identical. This conclusion does not requires the finiteness of v nor the isomorphy of $G_{\restriction K}$ and $G'_{\restriction K}$, it only requires that $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges for all k-element subsets K of V, simply because the adjacency matrix of the Kneser graph $KG(2, k+2)$ is non-singular (see Section 2).

In this paper we look for similar results if the conditions on the restrictions $G_{\restriction K}$ and $G'_{\restriction K}$ are given up to complementation, that is if $G'_{\restriction K}$ is isomorphic to $G_{\restriction K}$ or to its complement $\overline{G}_{\restriction K}$, or if $G'_{\restriction K}$ has the same number of edges than $G_{\restriction K}$ or $\overline{G}_{\restriction K}$. If the first condition holds for all k-element subsets K of V, we say that G and G' are *k-hypomorphic up to complementation* and, if the second holds, we say that G and G' have *the same number of edges up to complementation*. We say that G is *k-reconstructible up to complementation* if every graph G', k-hypomorphic to G up to complementation, is isomorphic to G or its complement.

We show first that the equality of the number of edges, up to complementation, for the k-vertices induced subgraphs suffices for the equality up to complementation provided that $4 \leq k \neq 7$ and v is large enough (Theorem 2.15). Our proof is based on Ramsey's theorem for pairs [13].

Next, we give partial description of the set $\mathcal{S}$ of pairs (v,k) such that two graphs G and G' on the same set of v vertices are equal up to complementation whenever they are k-hypomorphic up to complementation.

Theorem 1.1 *1. Let $v \leq 2$, then $(v,k) \in \mathcal{S}$ iff $k \in \mathbb{N}$.*

2. Let $v > 2$ then $(v,k) \in \mathcal{S}$ implies $4 \leq k \leq v-2$.

(a) If $v \equiv 2 \ (mod\ 4)$, $(v,k) \in \mathcal{S}$ iff $4 \leq k \leq v-2$;

(b) If $v \equiv 0 \ (mod\ 4)$ or $v \equiv 3 \ (mod\ 4)$ then $(v,k) \in \mathcal{S}$ implies $k \leq v-3$ for infinitely many v and $4 \leq k \leq v-3$ implies $(v,k) \in \mathcal{S}$;

(c) If $v \equiv 1 \ (mod\ 4)$ then $(v,k) \in \mathcal{S}$ implies $k \leq v-4$ for infinitely many v and $4 \leq k \leq v-4$ implies $(v,k) \in \mathcal{S}$.

Our proof for membership in $\mathcal{S}$ is a straithforward application of properties of incidence matrices due to D.H. Gottlieb [6], W. Kantor [9] and R.M. Wilson [16]. It is given in Section 3. Constraints on $\mathcal{S}$ are given in Section 4.

Our motivation comes from the following problem raised by P. Ille: find the least integer k such that every graph G having a large number v of vertices is k-reconstructible up to complementation. With Theorem 1.1 we show that $k = 4$ (see Section 2).

A quite similar problem was raised by J.G. Hagendorf (1992) and solved by J.G. Hagendorf and G. Lopez [7]. Instead of graphs, they consider binary relations and instead of the complement of a graph, they consider the *dual* R^* of a binary relation R (where $(x,y) \in R^*$ if and only if $(y,x) \in R$); they prove that 12 is the least integer k such that two binary relations R and R', on the same large set of vertices, are either isomorphic or dually isomorphic provided that the restrictions $R_{\restriction K}$ and $R'_{\restriction K}$ are isomorphic or dually isomorphic, for every k-element subsets K of V.

2 Preliminaries

Our notations and terminology follow [1]. A *graph* is a pair $G := (V, \mathcal{E})$, where $\mathcal{E}$ is a subset of $[V]^2$, the set of pairs $\{x, y\}$ of distinct elements of V. Elements of V are the *vertices* of G and elements of $\mathcal{E}$ its *edges*. If K is a subset of V, the *restriction* of G to K, also called the *induced graph* on K is the graph $G_{\restriction K} := (K, [K]^2 \cap \mathcal{E})$. If $K = V \setminus \{x\}$, we denote this graph by G_{-x}. The *complement* of G is the graph $\overline{G} := (V, [V]^2 \setminus \mathcal{E})$. We denote by $V(G)$ the vertex set of a graph G, by $E(G)$ its edge set and by $e(G) := |E(G)|$ the number of edges. If $\{x, y\}$ is an edge of G we set $G(x, y) = 1$; otherwise we set $G(x, y) = 0$. The *degree* of a vertex x of G, denoted $d_G(x)$, is the number of edges which contain x. The graph G is *regular* if $d_G(x) = d_G(y)$ for all $x, y \in V$. If G, G' are two graphs, we denote by $G \simeq G'$ the fact that they are isomorphic. A graph is *self-complementary* if it is isomorphic to its complement.

2.1 Incidence matrices and isomorphy up to complementation

Let V be a finite set, with v elements. Given non-negative integers t, k, let $W_{t\,k}$ be the $\binom{v}{t}$ by $\binom{v}{k}$ matrix of 0's and 1's, the rows of which are indexed by the t-element subsets T of V, the columns are indexed by the k-element subsets K of V, and where the entry $W_{t\,k}(T, K)$ is 1 if $T \subseteq K$ and is 0 otherwise.

A fundamental result, due to D.H. Gottlieb [6], and independently W. Kantor [9], is this:

Theorem 2.1 *For $t \leq min(k, v-k)$, $W_{t\,k}$ has full row rank over the field $\mathbb{Q}$ of rational numbers.*

If $k := v - t$ then, up to a relabelling, $W_{t\,k}$ is the adjacency matrix $A_{t,v}$ of the *Kneser graph* $KG(t, v)$, graph whose vertices are the t-element subsets of V, two subsets forming an edge if there are disjoint.

An equivalent form of Theorem 2.1 is:

Theorem 2.2 *$A_{t,v}$ is non-singular for $t \leq \frac{v}{2}$.*

Applications to graphs and relational structures where given in [5] and [11].

Theorem 2.1 has a modular version due to R.M. Wilson [16].

Theorem 2.3 *For $t \leq min(k, v-k)$, the rank of $W_{t\,k}$ modulo a prime p is*

$$\sum \binom{v}{i} - \binom{v}{i-1}$$

where the sum is extended over those indices i such that p does not divide the binomial coefficient $\binom{k-i}{t-i}$.

In the statement of the theorem, $\binom{v}{-1}$ should be interpreted as zero.

We will apply Wilson's theorem with $t = p = 2$ for $k \equiv 0 \ (mod\ 4)$ and for $k \equiv 1 \ (mod\ 4)$. In the first case the rank of $W_{2\,k}$ $(mod\ 2)$ is $\binom{v}{2} - 1$. In the second case, the rank is $\binom{v}{2} - v$.

Let us explain why the use of these results in our context is natural.

Let $X_1, \cdots, X_r$ be an enumeration of the 2-element subsets of V; let $K_1, \cdots, K_s$ be an enumeration of the k-element subsets of V and $W_{2\,k}$ be the matrix of the 2-element subsets versus the k-element subsets. If G is a graph with vertex set V, let w_G be the row

matrix $(g_1, \cdots, g_r)$ where $g_i = 1$ if X_i is an edge of G, 0 otherwise. We have $w_G W_{2\,k} = (e(G_{\restriction K_1}), \cdots, e(G_{\restriction K_s}))$. Thus, if G and G' are two graphs with vertex set V such that $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges for every k-element subset of V, we have $(w_G - w_{G'})W_{2\,k} = 0$. Thus, provided that $v \geq 4$, by Theorem 2.1, $w_G - w_{G'} = 0$ that is $G = G'$.

This proves the observation made at the beginning of our introduction. The same line of proof gives:

Proposition 2.4 *Let $t \leq min(v, v-k)$ and G and G' be two graphs on the same set V of v vertices. If G and G' are k-hypomorphic up to complementation then they are t-hypomorphic up to complementation.*

Proof. Let H be a graph on l vertices. Set $Is(H,G) := \{L \subseteq V : G_{\restriction L} \simeq H\}$, $Isc(H,G) := Is(H,G) \cup Is(\overline{H}, G)$ and $w_{H,G}$ the $0-1$-row vector indexed by the t-element subsets $X_1, \cdots, X_r$ of V whose coefficient of X_i is 1 if $X_i \in Isc(H,G)$ and 0 otherwise. From our hypothesis, it follows that $w_{H,G}W_{t\,k} = w_{H,G'}W_{t\,k}$. From Theorem 2.1, this implies $w_{H,G} = w_{H,G'}$ that is $Isc(H,G) = Isc(H,G')$. Since this equality holds for all graphs H on t-vertices, the conclusion of the lemma follows. □

Now, let $4 \leq k \leq v-4$ and G, G' be two graphs on the same set V of v vertices which are k-hypomorphic up to complementation. Then, as shown by Proposition 2.4 these two graphs are 4-hypomorphic up to complementation. By a carefull case analysis (or a very special case of Wilson's theorem, see Theorem 2.6 below), one can prove that two graphs on 6 vertices which are 4-hypomorphic up to complementation are in fact equal up to complementation. Hence,

Theorem 2.5 *$(k, v) \in \mathcal{S}$ for all v, k such that $4 \leq k \leq v-4$.*

P.Ille [8] asked for the least integer k such that every graph G having a large number v of vertices is k-reconstructible up to complementation.

From Theorem 2.5 above, k exists and is at most 4. From Proposition 4.1 below, we have $k \geq 4$. Hence $k = 4$.

This was our original solution of Ille's problem. The use of Wilson's theorem leads to the improvement of Theorem 2.5 contained in Theorem 1.1. Its use is natural too. Indeed, if we look at conditions which imply $G' = G$ or $G' = \overline{G}$, it is simpler to consider the *boolean sum* $G \dot{+} G'$ of G and G', that is the graph U on V whose edges are pairs e of vertices such that $e \in E(G)$ if and only if $e \notin E(G')$. Indeed, $G' = G$ or $G' = \overline{G}$ amounts to the fact that U is either the empty graph or the complete graph. But then, the use of $W_{2\,k}$ $(mod\ 2)$ become natural. Particularly, if conditions which insure $w_U W_{2\,k} = (0, \cdots, 0)$ $(mod\ 2)$ yield to $w_U = (0, \cdots, 0)$ or $w_U = (1, \cdots, 1)$, that is U is empty or complete, so $G' = G$ or $G' = \overline{G}$.

For example, we show first that if the parity of $e(G_{\restriction K})$ is the same than $e(G'_{\restriction K})$ for all k-element subsets K of V, then this may suffice to obtain $G' = G$ or $G' = \overline{G}$.

Theorem 2.6 *Let G and G' be two graphs on the same set V of v vertices (possibly infinite). Let k be an integer such that $4 \leq k \leq v-2$, $k \equiv 0\ (mod\ 4)$. Then the following properties are equivalent:*
(i) $e(G_{\restriction K})$ has the same parity than $e(G'_{\restriction K})$ for all k-element subsets K of V;
(ii) $G' = G$ or $G' = \overline{G}$.

Proof.

The implication $(ii) \Rightarrow (i)$ is trivial. We prove $(i) \Rightarrow (ii)$.

Lemma 2.7 *If $k \equiv 0$ (mod 4) or $k \equiv 1$ (mod 4), then under condition (i), $e(U_{\restriction K}) \equiv 0$ (mod 2) for all k-element subsets K of V.*

Proof. We have trivially :

Claim 2.8 *Let G be a graph of k vertices, then $e(G) + e(\overline{G})$ is even iff $k \equiv 0$ (mod 4) or $k \equiv 1$ (mod 4).*

Claim 2.9 *Let G and G' be two graphs on the same k-element vertex set V and let $G \cap G' := (V, E(G) \cap E(G'))$, then :*

$$e(G \dot{+} G') = e(G) + e(G') - 2e(G \cap G')$$

From this, we get :

Claim 2.10 *Let G and G' be two graphs on a k-element vertex set V, $k \equiv 0$ (mod 4) or $k \equiv 1$ (mod 4), and let $U := G \dot{+} G'$. If $e(G') \equiv e(G)$ (mod 2) or $e(G') + e(G) \equiv 0$ (mod 2) then $e(U) \equiv 0$ (mod 2).*

The conclusion of Lemma 2.7 follows. □

In order to prove implication $(i) \Rightarrow (ii)$ we may suppose V finite. With the notations above, we have $w_U W_{2\,k} = (e(U_{\restriction K_1}), \cdots, e(U_{\restriction K_s}))$. Thus, by Lemma 2.7, $w_U W_{2\,k} = (0, \cdots, 0)$ modulo 2. Since by Wilson's theorem, the rank of $W_{2\,k}$ modulo 2 is $\binom{v}{2} - 1$, the kernel of its transpose ${}^t W_{2\,k}$ has dimension 1. Since $(1, \cdots, 1) W_{2\,k} = (0, \cdots, 0)$ (mod 2) then $w_U W_{2\,k} = (0, \cdots, 0)$ (mod 2) amounts to $w_U = (0, \cdots, 0)$ or $w_U = (1, \cdots, 1)$, that is U is empty or complete, so $G' = G$ or $G' = \overline{G}$. □

Remark 2.11 *For every integer $k \not\equiv 0$ (mod 4) there are two graphs G and G' on the same vertex set V, $|V| \geq k + 2$, such that $e(G_{\restriction K})$ has the same parity than $e(G'_{\restriction K})$ or $\frac{k(k-1)}{2} - e(G'_{\restriction K})$ for all k-element subsets K of V, but G and G' are not isomorphic up to complementation.*

For an example, consider G and G' on the same vertex set $V := \{1, \cdots, v\}$ such that the edges of G' are $(1, i)$ for all $i \in \{2, 3, \cdots, v\}$ and G is the empty graph if $k \equiv 3$ (mod 4) or $k \equiv 1$ (mod 4); G is a complete graph if $k \equiv 2$ (mod 4).

We give an analog of Theorem 2.6 in the case $k \equiv 1$ (mod 4). For that, an additional condition is needed.

Let G be a graph. A 3-element subset T of V such that all pairs belong to $E(G)$ is a *triangle* of G. A 3-element subset of V which is a triangle of G or of $\overline{G}$ is a 3-*homogeneous* subset of G.

Theorem 2.12 *Let G and G' be two graphs on the same set V of v vertices (possibly infinite). Let k be an integer such that $5 \leq k \leq v - 2$, $k \equiv 1$ (mod 4). Then the following properties are equivalent:*
(i) $e(G_{\restriction K})$ has the same parity than $e(G'_{\restriction K})$ for all k-element subsets K of V and the same 3-homogeneous subsets;
(ii) $G' = G$ or $G' = \overline{G}$.

Proof. The implication $(ii) \Rightarrow (i)$ is trivial. We prove $(i) \Rightarrow (ii)$.
We may suppose V finite. Let $U := G\dot{+}G'$. From the fact that $e(G_{\restriction K})$ has the same parity than $e(G'_{\restriction K})$ for all k-element subsets K, the boolean sum U belongs to the kernel of ${}^tW_{2\,k}$ (over the 2-element field).

Claim 2.13 *Let k be an integer such that $2 \leq k \leq v-2$, $k \equiv 1$ $(mod\ 4)$, then the kernel of ${}^tW_{2\,k}$ consists of complete bipartite graphs and their complements (including the empty graph and the complete graph).*

Proof. Let us recall that a *star-graph* of v vertices consists of a vertex linked to all other vertices, those $v-1$ vertices forming an independent set. The vector space (over the 2-element field) generated by the star-graphs on V consists of all complete bipartite graphs distinct from the complete graph (but including the empty graph). Moreover, its dimension is $v-1$ (a basis being made of star-graphs). Let $\mathbb{K}$ be the kernel of ${}^tW_{2\,k}$. Since k is odd, each star-graph belongs to $\mathbb{K}$. Since $k \equiv 1$ $(mod\ 4)$, the complete graph also belongs to $\mathbb{K}$. According to Wilson's theorem, the rank of W_{2k} $(mod\ 2)$ is $\binom{v}{2} - v$. Hence the kernel of ${}^tW_{2\,k}$ has dimension v. Consequently, $\mathbb{K}$ consists of complete bipartite graphs and their complements, as claimed. □

A *claw* is a star-graph on four vertices, that is a graph made of a vertex joined to three other vertices, with no edges between these three vertices. A graph is *claw-free* if no induced subgraph is a claw.

Claim 2.14 *Let G and G' be two graphs on the same set and having the same 3-homogeneous subsets, then the boolean sum $U := G\dot{+}G'$ and its complement are claw-free.*

Proof. Let $x \in V$. Suppose $d_U(x) \geq 3$. Then, the neighborhood of x contains at least two distinct vertices y, y' such that $U(y, y') = 1$. Indeed, it contains clearly two vertices y, y' such that $G(x, y) = G(x, y')$. If $U(y, y') = 0$, that is $G(y, y') = G'(y, y')$, then since G and G' have the same 3-element homogeneous sets and $G(x, y) \neq G'(x, y)$, $\{x, y, y'\}$ cannot be homogeneous, hence $G(y, y') \neq G(x, y)$ and $G'(y, y') \neq G'(x, y)$. This implies $G(y, y') \neq G'(y, y')$, a contradiction. From this observation, U is claw-free. Since G and $\overline{G}'$ have the same 3-homogeneous subsets and $\overline{U} = G\dot{+}\overline{G}'$, we also get that $\overline{U}$ is claw-free.
□

For a characterization of these boolean sums, see [12].

From Claim 2.13, U or its complement is a complete bipartite graph and, from Claim 2.14, U and $\overline{U}$ are claw-free. Since $v \geq 5$ (in fact $v \geq 7$), it follows that U is either the empty graph or the complete graph. Hence $G' = G$ or $G' = \overline{G}$ as claimed. □

2.2 Conditions on the number of edges and Ramsey's theorem

Theorem 2.15 *Let k be an integer, $7 \neq k \geq 4$. There is an integer m such that if G and G' are two graphs on the same set V of v vertices, $v \geq m$, such that $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, for all k-element subsets K of V, then $G' = G$ or $G' = \overline{G}$.*

Conditions $7 \neq k \geq 4$ in Theorem 2.15 are necessary.

- For $k = 7$, consider two graphs G and G' on $V := \{1, 2, \cdots, v\}$ such that $\{i, j\}$ is an edge of G and G' for all $i \neq j$ in $\{1, 2, \cdots, v-2\}$, G has no another edge and G' has $\{v-1, v\}$ as an additional edge. For $k \geq 4$ apply Proposition 4.1 below.

Let $c(k)$ be the least integer m for which the conclusion of Theorem 2.15 holds.

Problem 2.16 *Is* $c(k) \leq k+4$*?*

Our proof uses Ramsey's theorem rather than incidence matrices. It is inspired from a relationship between Ramsey's theorem and Theorem 2.1 pointed out in [11]. The drawback is that the bound on $c(k)$ is quite crude.

Let $r_2^2(k)$ be the bicolor Ramsey number for pairs: the least integer n such that every graph on n vertices contains a k-homogeneous subset, that is a clique or an independent on k vertices. We deduce Theorem 2.15 and $c(k) \leq r_2^2(k)$ from the following result.

Proposition 2.17 *Let k be an integer, $7 \neq k \geq 4$ and let G and G' be two graphs on the same set V of v vertices, $v \geq k$ such that:*

1. *$G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, for all k-element subsets K of V;*
2. *V contains a k-element subset K such that $G_{\restriction K}$ or $\overline{G}_{\restriction K}$ has at least l edges where $l := \min\left(\frac{k^2+7k-12}{4}, \frac{k(k-1)}{2}\right)$.*

Then $G' = G$ or $G' = \overline{G}$.

The inequality $\frac{k^2+7k-12}{4} \leq \frac{k(k-1)}{2}$ holds iff $k \geq 8$. For $k > 8$ the condition $l = \frac{k^2+7k-12}{4}$ is weaker than the existence of a clique of size k.

Proof. We may suppose that V contains a k-element subset of V, say K, such that $e(G_{\restriction K}) \geq l$; also we may suppose, from condition 1, that $e(G_{\restriction K}) = e(G'_{\restriction K})$ otherwise replace G' by its complement. We shall prove that for all V' such that $K \subseteq V' \subseteq V$ and $\mid V' \mid = k+2$ we have $e(G_{\restriction K'}) = e(G'_{\restriction K'})$ for all k-element subset K' of V'. Since the adjacency matrix of the Kneser graph $KG(2, k+2)$ is non-singular, $G_{\restriction V'} = G'_{\restriction V'}$. It follows that $G = G'$.

Claim 2.18 *For $x \notin K$ and $y \in K$, $e(G_{\restriction (K\cup\{x\})\setminus\{y\}}) = e(G'_{\restriction (K\cup\{x\})\setminus\{y\}})$.*

Proof. Let $x \notin K$ and $y \in K$. Set $K' := (K \cup \{x\}) \setminus \{y\}$. The graphs $G_{\restriction K'}$ and $G'_{\restriction K'}$ have at least $l' := l - (k-1)$ edges. Since $G_{\restriction K'}$ and $G'_{\restriction K'}$ have the same number of edges up to complementation, we have $e(G_{\restriction K'}) = e(G'_{\restriction K'})$ whenever $l' \geq \frac{k(k-1)}{4}$, that is $l \geq l'' := \frac{(k-1)(k+4)}{4}$.

If $k \geq 8$ we have $l = \frac{k^2+7k-12}{4}$ yielding $l > l''$ as required. If $k \in \{4,5,6\}$ we have $l = \frac{k(k-1)}{2}$ yielding again $l \geq l''$. □

Claim 2.19 *For distinct $x, x' \notin K$ and $y, y' \in K$, $e(G_{\restriction (K\cup\{x,x'\})\setminus\{y,y'\}}) = e(G'_{\restriction (K\cup\{x,x'\})\setminus\{y,y'\}})$.*

Proof. Let $x, x' \notin K$ and $y, y' \in K$ be distinct. Set $K' := (K \cup \{x, x'\}) \setminus \{y, y'\}$. We have $e(G_{\restriction K'}) \geq e(G_{\restriction K}) - (2k-3)$ and $e(G'_{\restriction K'}) \geq e(G_{\restriction K}) - (2k-3)$. Thus $e(G_{\restriction K'})$ and $e(G'_{\restriction K'})$ have at least $l' := l - (2k-3)$ edges. Since $G_{\restriction K'}$ and $G'_{\restriction K'}$ have the same number of edges up to complementation, we have $e(G_{\restriction K'}) = e(G'_{\restriction K'})$ whenever $l' \geq \frac{k(k-1)}{4}$, that is $l \geq \frac{k^2+7k-12}{4}$. This inequality holds if $k \geq 8$.

Suppose $k \in \{4,5,6\}$. Thus $l = \frac{k(k-1)}{2}$. Hence K is a clique for G and G'.

Subclaim Let $u \notin K$ then G and G' coincide on $K \cup \{u\}$.

Proof. Since K is a clique, this amounts to $G(u,v) = G'(u,v)$ for all $v \in K$, a fact which follows from Claim 2.18. Indeed, we have $d_{G_{\restriction K\cup\{u\}}}(u) = \frac{1}{k-1}\sum_{w\in K} d_{G_{\restriction (K\cup\{u\})\setminus\{w\}}}(u)$. From

Claim 2.18 we have $d_{G_{\restriction (K\cup\{u\})\setminus\{w\}}}(u) = d_{G'_{\restriction (K\cup\{u\})\setminus\{w\}}}(u)$. Thus $d_{G_{\restriction K\cup\{u\}}}(u) = d_{G'_{\restriction K\cup\{u\}}}(u)$. Since $d_{G_{\restriction (K\cup\{u\})\setminus\{v\}}}(u) = d_{G'_{\restriction (K\cup\{u\})\setminus\{v\}}}(u)$ the equality $G(u,v) = G'(u,v)$ follows. $\square$

From this subclaim it follows that G and G' coincide on K' with the possible exception of the pair $\{x, x'\}$. Set $a := e(G_{\restriction K'})$, $a' := e(G'_{\restriction K'})$. Suppose $a \neq a'$. Then $|a - a'| = 1$, hence the sum $a + a'$ is odd. Since $G_{\restriction K'}$ and $G'_{\restriction K'}$ have the same number of edges up to complementation, this sum is also $\frac{k(k-1)}{2}$. If $k = 4$ or $k = 5$ this number is even, a contradiction. Suppose $k = 6$. We may suppose $a = a' + 1$ hence from $a + a' = \frac{k(k-1)}{2}$ we get $a = 8$. Put $\{x_1, x_2, x_3, x_4, y, y'\} := K$. Since K is a clique we have $G(x, x') = 1$, $G'(x, x') = 0$ and G, G' contain just an edge from $\{x, x'\}$ to $\{x_1, x_2, x_3, x_4\}$. We may suppose $G(x_1, x) = G'(x_1, x) = 1$, $G(x_1, x') = G'(x_1, x') = 0$ and $G(t, u) = G'(t, u) = 0$ for all $t \in \{x_2, x_3, x_4\}$ and $u \in \{x, x'\}$.
Let $K'' := (K \cup \{x, x'\}) \setminus \{x_1, x_2\}$. From the subclaim above, G and G' coincide on K'' at the exception of the pair $\{x, x'\}$ hence G, G' contain just an edge from $\{x, x'\}$ to $\{x_3, x_4, y, y'\}$. We can assume $G(y, u) = G'(y, u) = 1$ for exactly one $u \in \{x, x'\}$, and $G(t, u) = G'(t, u) = 0$ for all $t \in \{x_3, x_4, y'\}$ and $u \in \{x, x'\}$.

Set $B := \{x_2, x_3, x_4, x, x', y'\}$, then $e(G_{\restriction B}) = 7$ and $e(G'_{\restriction B}) = 6$. So $e(G_{\restriction B}) \neq e(G'_{\restriction B})$ and $e(G_{\restriction B}) + e(G'_{\restriction B}) \neq \frac{k(k-1)}{2}$, that gives a contradiction. $\square$
Clearly Proposition 2.17 follows from Claims 2.18 and 2.19. $\square$

3 Some members of $\mathcal{S}$

Sufficient conditions for membership stated in Theorem 1.1 are contained in Theorem 3.1 below.

Let v be a non negative integer and $\vartheta(v) := 4l$ if $v \in \{4l + 2, 4l + 3\}$, $\vartheta(v) := 4l - 3$ if $n \in \{4l, 4l + 1\}$.

Theorem 3.1 *Let v, k be two integers , $v \geq 6$, $4 \leq k \leq \vartheta(v)$. Then, for every pair of graphs G and G' on the the same set V of v vertices, the following properties are equivalent:*

(i) G and G' are k-hypomorphic up to complementation;

(ii) $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, and the same number of 3-homogeneous subsets, for all k-element subsets K of V;

(iii) $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges, up to complementation, for all k-element and k'-element subsets K of V where k' is an integer verifying $3 \leq k' < k$;

(iv) $G' = G$ or $G' = \overline{G}$.

3.1 Ingredients

Let $G := (V, E)$ be a graph. Let $A^{(2)}(G)$ be the set of unordered pairs $\{u, u'\}$ made of some $u \in E(G)$ and some $u' \in E(\overline{G})$. Let $A^{(0)}(G) := \{\{u, u'\} \in A^{(2)}(G) \ : \ u \cap u' = \emptyset\}$, $A^{(1)}(G) := A^{(2)}(G) \setminus A^{(0)}(G)$ and let $a^{(i)}(G)$ be the cardinality of $A^{(i)}(G)$ for $i \in \{0, 1, 2\}$; thus $a^{(2)}(G) = a^{(0)}(G) + a^{(1)}(G)$. Let $T(G)$ be the set of *triangles* of G and let $t(G) :=\mid T(G) \mid$. Let $H^{(3)}(G) := T(G) \cup T(\overline{G})$ be the set of 3-homogeneous subsets of G and $h^{(3)}(G) :=\mid H^{(3)}(G) \mid$.

Some elementary properties of the above numbers are stated in the lemma below; the proof is immediate.

Lemma 3.2 *Let G be a graph with v vertices, then :*
1) $A^{(i)}(G) = A^{(i)}(\overline{G})$, hence $a^{(i)}(G) = a^{(i)}(\overline{G})$, for all $i \in \{0,1,2\}$.
2) $a^{(2)}(G) = e(G)e(\overline{G})$.
3) $a^{(1)}(G) = \sum_{x \in V(G)} d_G(x) d_{\overline{G}}(x)$.
4) $h^{(3)}(G) = \frac{v(v-1)(v-2)}{6} - \frac{1}{2} a^{(1)}(G)$.

Lemma 3.3 *Let G and G' be two graphs on the same finite vertex set V, then :*

$$e(G') = e(G) \text{ or } e(G') = e(\overline{G}) \text{ iff } e(G)e(\overline{G}) = e(G')e(\overline{G'})$$

Proof. Suppose :

$$e(G)e(\overline{G}) = e(G')e(\overline{G'}) \tag{1}$$

Since $e(G) + e(\overline{G}) = \frac{v(v-1)}{2}$ and $e(G') + e(\overline{G'}) = \frac{v(v-1)}{2}$, where $v :=\mid V \mid$, we have :

$$e(G) + e(\overline{G}) = e(G') + e(\overline{G'}) \tag{2}$$

Then (1) and (2) give $e(G') = e(G)$ *or* $e(G') = e(\overline{G})$. The converse is obvious. □

Lemma 3.4 *Let G be a graph, $V := V(G), v :=\mid V \mid$.*
a) Let $i \in \{0,1\}$, k such that $4 - i \leq k \leq v$, then :

$$a^{(i)}(G) = \frac{1}{\binom{v-4+i}{k-4+i}} \sum_{\substack{K \subseteq V \\ |K|=k}} a^{(i)}(G_{\restriction K})$$

b) Let k such that $3 \leq k \leq v-1$, then :

$$a^{(0)}(G) = \frac{v-3}{v-k} e(G)e(\overline{G}) - \frac{1}{\binom{v-4}{k-3}} \sum_{\substack{K \subseteq V \\ |K|=k}} e(G_{\restriction K})e(\overline{G}_{\restriction K})$$

$$a^{(1)}(G) = \frac{1}{\binom{v-4}{k-3}} \sum_{\substack{K \subseteq V \\ |K|=k}} e(G_{\restriction K})e(\overline{G}_{\restriction K}) - \frac{k-3}{v-k} e(G)e(\overline{G})$$

Proof. a) Let $\{u, u'\} \in A^{(i)}(G)$ for $i \in \{0,1\}$. The number of k-element subsets K of V containing u and u' is $\binom{v-4+i}{k-4+i}$. Then we conclude.

b) If $k = 3$ then a) and the fact that $a^{(0)}(G) + a^{(1)}(G) = e(G)e(\overline{G})$ give the formulas. If $4 \leq k \leq v-1$, then by a) we have :

$$\binom{v-4}{k-4} a^{(0)}(G) = \sum_{\substack{K \subseteq V \\ |K|=k}} a^{(0)}(G_{\restriction K})$$

$$\binom{v-3}{k-3} a^{(1)}(G) = \sum_{\substack{K \subseteq V \\ |K|=k}} a^{(1)}(G_{\restriction K})$$

Summing up and applying 2) of Lemma 3.2 to the $G_{\restriction K}$'s we have :

$$\binom{v-4}{k-4} a^{(0)}(G) + \binom{v-3}{k-3} a^{(1)}(G) = \sum_{\substack{K \subseteq V \\ |K|=k}} e(G_{\restriction K})e(\overline{G}_{\restriction K}) \tag{3}$$

On an other hand :

$$a^{(0)}(G) + a^{(1)}(G) = e(G)e(\overline{G}) \tag{4}$$

Equations (3) and (4) form a Cramer system with $a^{(0)}(G)$ and $a^{(1)}(G)$ as unknowns. Indeed the determinant $\Delta := \begin{vmatrix} \binom{v-4}{k-4} & \binom{v-3}{k-3} \\ 1 & 1 \end{vmatrix} = \binom{v-4}{k-4} - \binom{v-3}{k-3} = -\binom{v-4}{k-3}$ is non zero. A straightforward computation gives the result. □

Corollary 3.5 *Let G and G' be two graphs on the same set V of v vertices and k be an integer such that $4 \leq k \leq v$.*

The implications $(ii) \Rightarrow (i)$ and $(i) \Rightarrow (iii)$ between the following statements hold.

(i) $e(G'_{\restriction K}) = e(G_{\restriction K})$ *or* $e(\overline{G}_{\restriction K})$ *and* $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ *for all k-element subsets K of V.*

(ii) $e(G'_{\restriction K}) = e(G_{\restriction K})$ *or* $e(\overline{G}_{\restriction K})$ *for all k-element and k'-element subsets K of V where k' is some integer verifying $3 \leq k' < k$.*

(iii) $G_{\restriction L}$ *and* $G'_{\restriction L}$ *have the same number of edges up to complementation and* $h^{(3)}(G_{\restriction L}) = h^{(3)}(G'_{\restriction L})$ *for all l-element subsets L of V and all integer l such that $k \leq l \leq v$.*

Proof. $(i) \Rightarrow (iii)$. Let L be an l-element subset of V with $l \geq k$, and K be a k-element subset of L. From Lemma 3.3 and 2) of Lemma 3.2, we have $a^{(0)}(G_{\restriction K}) + a^{(1)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K}) + a^{(1)}(G'_{\restriction K})$, and from 4) of Lemma 3.2, $a^{(1)}(G_{\restriction K}) = a^{(1)}(G'_{\restriction K})$. Hence $a^{(i)}(G_{\restriction K}) = a^{(i)}(G'_{\restriction K})$ for all k-element subsets K of L and $i \in \{0, 1\}$.
From a) of Lemma 3.4 applied to $G_{\restriction L}$ follows $a^{(i)}(G_{\restriction L}) = a^{(i)}(G'_{\restriction L})$ *for* $i \in \{0, 1\}$, hence using 2) of Lemma 3.2 we get $e(G_{\restriction L})e(\overline{G}_{\restriction L}) = e(G'_{\restriction L})e(\overline{G'}_{\restriction L})$. The conclusion follows from Lemma 3.3 and 4) of Lemma 3.2.

$(ii) \Rightarrow (i)$. It suffices to prove that $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ for all k-element subsets K of V. From Lemma 3.3 we have $e(G_{\restriction K})e(\overline{G}_{\restriction K}) = e(G'_{\restriction K})e(\overline{G}'_{\restriction K})$ and $e(G_{\restriction K'})e(\overline{G}_{\restriction K'}) = e(G'_{\restriction K'})e(\overline{G}'_{\restriction K'})$ for all k'-element set $K' \subseteq K$. From b) of Lemma 3.4 we get $a^{(i)}(G_{\restriction K}) = a^{(i)}(G'_{\restriction K})$ for $i \in \{0, 1\}$. Then by 4) of Lemma 3.2, $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$. □

Proposition 3.6 *Let G and G' be two graphs on v vertices and k be an integer such that $4 \leq k \leq v$. If G and G' are k-hypomorphic up to complementation then $e(G'_{\restriction L}) = e(G_{\restriction L})$ or $e(G'_{\restriction L}) = e(\overline{G}_{\restriction L})$ for all l-element subsets L of V and all integer l such that $k \leq l \leq v$.*

Proof. If G and G' are k-hypomorphic up to complementation then $G_{\restriction K}$ and $G'_{\restriction K}$ have the same number of edges up to complementation, and the same number of 3-homogeneous subsets, for all k-element subsets K of V. We conclude using $(i) \Rightarrow (iii)$ of Corollary 3.5 □

By inspection of the eleven graphs on four vertices, one may observe that:

Fact 3.7 *The pair $(e(G)e(\overline{G}), h^{(3)}(G))$ characterize G up to isomorphy and complementation if $\mid V(G) \mid \leq 4$.*

Note that in Fact 3.7, we can replace $(e(G)e(\overline{G}), h^{(3)}(G))$ by $(a^{(0)}(G), a^{(1)}(G))$ (this follows from Lemmas 3.3 and 3.2).

Proposition 3.8 *Let G and G' be two graphs on the same set V of v vertices and k be an integer. If $3 \le k \le v-3$ (resp. $4 \le k \le v-4$) and $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ (resp. $a^{(0)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K})$) for all k-element subsets K of V then $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ (resp. $a^{(0)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K})$) for all $(v-k)$-element subsets K of V.*

Proof. By 4) of Lemma 3.2, $h^{(3)}(G_{\restriction K}) = h^{(3)}(G'_{\restriction K})$ iff $a^{(1)}(G_{\restriction K}) = a^{(1)}(G'_{\restriction K})$.

Case 1. $k \le \frac{v}{2}$, then $v-k \ge k$. Let K' be a $(v-k)$-element subset of V, then from a) of Lemma 3.4 we have for $i \in \{0,1\}$:

$$a^{(i)}(G_{\restriction K'}) = \frac{1}{\binom{v-k-4+i}{k-4+i}} \sum_{\substack{K \subseteq K' \\ |K|=k}} a^{(i)}(G_{\restriction K})$$

Then we get the conclusion.

Case 2. $k > \frac{v}{2}$, then $v-k < \frac{v}{2}$. Let K' be a k-element subset of V. From a) of Lemma 3.4 we have for $i \in \{0,1\}$:

$$\sum_{\substack{K \subseteq K' \\ |K|=v-k}} a^{(i)}(G_{\restriction K}) = \binom{k-4+i}{v-k-4+i} a^{(i)}(G_{\restriction K'}) \tag{5}$$

Let $X_1, X_2, \cdots, X_l$ be an enumeration of the $(v-k)$-element subsets of V. Let $w_G^{(i)} := (a^{(i)}(G_{\restriction X_1}), a^{(i)}(G_{\restriction X_2}), \cdots, a^{(i)}(G_{\restriction X_l}))$, and $w_{G'}^{(i)} := (a^{(i)}(G'_{\restriction X_1}), a^{(i)}(G'_{\restriction X_2}), \cdots, a^{(i)}(G'_{\restriction X_l}))$. From (5), we get, for $i \in \{0,1\}$: $A_{v,v-k}{}^t w_G^{(i)} = A_{v,v-k}{}^t w_{G'}^{(i)}$. We conclude using Theorem 2.2. $\square$

3.2 Proof of Theorem 3.1.

$(i) \Rightarrow (ii)$, $(iv) \Rightarrow (i)$, $(iv) \Rightarrow (iii)$ are obvious and $(iii) \Rightarrow (ii)$ is implication $(ii) \Rightarrow (i)$ of Corollary 3.5. Thus it is sufficient to prove $(ii) \Rightarrow (iv)$.
Let l, $k \le l \le v$. According to implication $(i) \Rightarrow (ii)$ of Corollary 3.5, $e(G'_{\restriction L}) = e(G_{\restriction L})$ or $e(G'_{\restriction L}) = e(\overline{G}_{\restriction L})$ for all l-element subsets L of V. If we may choose $l \equiv 0 \ (mod\ 4)$ with $l \le v-2$, then from Claim 2.8, $e(G_{\restriction L})$ and $e(G'_{\restriction L})$ have the same parity. Theorem 2.6 gives $G' = G$ or $G' = \overline{G}$. Thus, the implication $(ii) \Rightarrow (iv)$ is proved if $v \equiv 2 \ (mod\ 4)$ and if $v \equiv 3 \ (mod\ 4)$. There are two remaining cases.

Case 1. $v \equiv 1 \ (mod\ 4)$ and $k = v-4$. We prove that $e(G'_{\restriction L})$ and $e(G_{\restriction L})$ have the same parity for all 4-element subsets L of V. Theorem 2.6 again gives $G' = G$ or $G' = \overline{G}$. The proof goes as follows. Let L be a 4-element subset of V, and K be a k-element subset of V. By Lemma 3.2, $a^{(2)}(G_{\restriction K}) = a^{(2)}(G'_{\restriction K})$ and $a^{(1)}(G_{\restriction K}) = a^{(1)}(G'_{\restriction K})$. Thus $a^{(0)}(G_{\restriction K}) = a^{(0)}(G'_{\restriction K})$. Using Proposition 3.8, we get $a^{(0)}(G_{\restriction L}) = a^{(0)}(G'_{\restriction L})$ and $h^{(3)}(G_{\restriction L}) = h^{(3)}(G'_{\restriction L})$. Now 4) of Lemma 3.2 gives $a^{(1)}(G_{\restriction L}) = a^{(1)}(G'_{\restriction L})$. So $a^{(2)}(G_{\restriction L}) = a^{(2)}(G'_{\restriction L})$, then using 2) of Lemma 3.2 and Lemma 3.3 we get $e(G'_{\restriction L}) = e(G_{\restriction L})$ or $e(\overline{G}_{\restriction L})$, thus $e(G'_{\restriction L})$ and $e(G_{\restriction L})$ have the same parity. $\square$

Case 2. $v \equiv 0 \ (mod\ 4)$ and $k = v-3$. From Proposition 3.8, G and G' have the same 3-homogeneous subsets. From Theorem 2.12, $G' = G$ or $G' = \overline{G}$ as claimed. $\square$

4 Constraints on $\mathcal{S}$

Two arbitrary graphs on the same set of vertices are k-hypomorphic up to complementation for $k \leq 2$. Hence, if $v \leq 2$, $(v, k) \in \mathcal{S}$ iff $k \in \mathbb{N}$. This is item 1 of Theorem 1.1.

Next, suppose $v > 2$, and $(v, k) \in \mathcal{S}$.

According to the proposition below, we have $k \geq 4$.

Proposition 4.1 *For every integer $v \geq 4$, there are two graphs G and G', on the same set of v vertices, which are 3-hypomorphic up to complementation but not isomorphic up to complementation.*

Proof. Let G and G' be two graphs having $\{1, 2, \cdots, v\}$ as set of vertices.
- Even case : $v = 2p$. Pairs $\{i, j\}$ are edges of G and G' for all $i \neq j$ in $\{1, 2, \cdots, p\}$ and for all $i \neq j$ in $\{p + 1, \cdots, 2p\}$. The graph G has no other edge and G' has $\{1, p + 1\}$ as an additional edge. Clearly G' and G are 3-hypomorphic up to complementation and not isomorphic. Since $\overline{G}$ has p^2 edges but G' has $p(p - 1) + 1$ edges, G' and $\overline{G}$ are not isomorphic.
- Odd case : $v = 2p+1$. Pairs $\{i, j\}$ are edges of G and G' for all $i \neq j$ in $\{1, 2, \cdots, p\}$ and for all $i \neq j$ in $\{p + 1, \cdots, 2p + 1\}$. The graph G has no other edge and G' has $\{1, p + 1\}$ as an additional edge. Clearly G' and G are 3-hypomorphic up to complementation and not isomorphic. Since $\overline{G}$ has $p(p + 1)$ edges but G' has $p^2 + 1$ edges, G' and $\overline{G}$ are not isomorphic.
In both cases G and G' are 3-hypomorphic up to complementation but not isomorphic up to complementation. □

According to the following lemma, $v \geq 6$.

Lemma 4.2 *For every v, $3 \leq v \leq 5$, there are two graphs G and G', on the same set of v vertices, which are k-hypomorphic for all $k \leq v$ but $G' \neq G$ and $G' \neq \overline{G}$.*

Proof. Let $V := \{0, 1, 2, 3, 4\}$, $\mathcal{E} := \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 0\}\}$, $\mathcal{E}' := (\mathcal{E} \setminus \{\{0, 4\}, \{1, 2\}\}) \cup \{\{1, 4\}, \{0, 2\}\}$. Let $G := (V, \mathcal{E})$ and $G' := (V, \mathcal{E}')$. These graphs are two 5-element cycles, G' being obtained from G by exchanging 0 and 1. Trivially, they satisfy the conclusion of the lemma. The two pairs G_{-3} , G'_{-3} and $G_{-3,-4}$ and $G'_{-3,-4}$ also satisfy the conclusion of the lemma. □

Next, a straightforward extension of the construction in Lemma 4 above yields $k \leq v-2$. Indeed, let us say that two graphs G and G' on the same set V of vertices are *k-hypomorphic* if for any subset X of V of cardinality k, $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic. We have:

Lemma 4.3 *For every integer v, $v \geq 4$, there are two graphs G and G', on the same set of v vertices, which are k-hypomorphic for all $k \in \{v - 1, v\}$ but $G' \neq G$ and $G' \neq \overline{G}$.*

Proof. Let $V := \{0, \ldots, v - 1\}$, $\mathcal{E} := \{\{i, i + 1\} : 0 \leq i < v - 1\} \cup \{\{0, v - 1\}\}$, $\mathcal{E}' := (\mathcal{E} \setminus \{\{0, v-1\}, \{1, 2\}\}) \cup \{\{1, v-1\}, \{0, 2\}\}$. Let $G := (V, \mathcal{E})$ and $G' := (V, \mathcal{E}')$. These graphs are two v-element cycles, G' being obtained from G by exchanging 0 and 1. Trivially, they satisfy the conclusion of the lemma. □

With this lemma, the proof of the first part of item 2 is complete.

The fact that $(v, k) \in \mathcal{S}$ implies $k \leq \vartheta(v)$ for infinitely many v is an immediate consequence of the following proposition.

Proposition 4.4 *For every integer $v := m + r$, where m is a product of prime powers p, $p \equiv 1 \pmod 4$, and $r \in \{2, 3, 4\}$ there are two graphs G and G', on the same set of v vertices, which are k-hypomorphic up to complementation for all k, $\vartheta(v) + 1 \leq k \leq v$ but $G' \neq G$ and $G' \neq \overline{G}$.*

Proof. Our construction relies on the following claim.

Claim 4.5 *For each integer m, where m is a product of prime powers p, $p \equiv 1$ $(mod\ 4)$, there is a graph P on m vertices such that P and P_{-x} are self complementary for every $x \in V(P)$.*

Let P be a graph satisfying the conclusion of Claim 4.5.

Case 1. $r := 4$. In this case $\vartheta(v) := m$. Let V be made of $V(P)$ and four new elements added, say $1, 2, 3, 4$. Let G and G' be the graphs with vertex set V which coincide with P on $V(P)$, the other edges of G being $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, x\}$, $\{3, x\}$ for all $x \in V(P)$, the other edges of G' being $\{1, 3\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, x\}$, $\{3, x\}$ for all $x \in V(P)$. Clearly, $G' \neq G$ and $G' \neq \overline{G}$. Let $X \subseteq V$ with $|X| \leq 3$ and $K := V \setminus X$. If $X \cap \{1, 2, 3, 4\} \in \{\{1, 3\}, \{2, 4\}\}$ then $G_{\restriction K} \simeq \overline{G'}_{\restriction K}$. In all other cases $G_{\restriction K} \simeq G'_{\restriction K}$. Hence G and G' are k-hypomorphic for $\vartheta(v) + 1 \leq k \leq v$.

Case 2. $r := 3$. In this case $\vartheta(v) := m$. Let $G_1 := G_{-1}$ and $G'_1 := G'_{-1}$ where G, G' are the graphs constructed in Case 1. Clearly $G' \neq G$ and $G' \neq \overline{G}$. And since G, G' are k-hypomorphic for $m + 1 \leq k \leq m + 4$, the graphs G_1 and G'_1 are k-hypomorphic for $\vartheta(v) + 1 \leq k \leq v$.

Case 3. $r := 2$. In this case $\vartheta(v) := m - 1$. Let V be made of $V(P)$ and two new elements added, say $1, 2$. Let G and G' be the graphs with vertex set V which coincide with P on $V(P)$, the other edges of G being $(2, x)$ for all $x \in V(P)$, the other edges of G' being $(1, x)$ for all $x \in V(P)$. Clearly, $G' \neq G$ and $G' \neq \overline{G}$. Let $X \subseteq V$ with $|X| \leq 2$ and $K := V \setminus X$. If $X \cap \{1, 2\} \neq \emptyset$ then $G_{\restriction K} \simeq \overline{G'}_{\restriction K}$. In all other cases $G_{\restriction K} \simeq G'_{\restriction K}$. Hence, G and G' are k-hypomorphic for $\vartheta(v) + 1 \leq k \leq v$. □

Claim 4.5 is an immediate consequence of the following result.

Lemma 4.6 *Let $\mathcal{G}$ be the class of finite graphs G of order distinct from 2 such that G_{-x} is self-complementary for every vertex $x \in V(G)$.*

1. *The class $\mathcal{G}$ coincides with the class of self-complementary vertex-transitive graphs.*
2. *The class $\mathcal{G}$ includes Paley graphs.*
3. *The class $\mathcal{G}$ is closed under lexicographic product.*

Proof.

1. Let $G \in \mathcal{G}$. Let $n := |V(G)|$. We may suppose $n > 2$. Let $x \in V(G)$. We have $d_G(x) = e(G) - e(G_{-x})$. Since G_{-x} is self-complementary, $e(G_{-x}) = e(\overline{G}_{-x})$ and, since $e(G_{-x}) + e(\overline{G}_{-x}) = \binom{n-1}{2}$, $e(G_{-x}) = \frac{1}{2}\binom{n-1}{2}$. Thus $d_G(x)$ does not depend on x, that is G is regular. Since $n > 2$ we have $e(G) = \frac{1}{n-2}\sum_{x \in V(G)} e(G_{-x})$ thus $e(G) = \frac{n(n-1)}{4}$. This added to $e(G_{-x}) = \frac{(n-1)(n-2)}{4}$ yields $n(n-1) \equiv 0$ $(mod\ 4)$ and $(n-1)(n-2) \equiv 0$ $(mod\ 4)$. It follows that $n \equiv 1$ $(mod\ 4)$. As it is well-known [10], regular graphs of order distinct from 2 are reconstructible. Thus G is self-complementary. The proof that G is reconstructible yields that for every vertex x, every isomorphism from G_{-x} onto $\overline{G}_{-x}$ is induced by an isomorphism φ from G onto $\overline{G}$ which fixes x. Hence, for a given pair of vertices x, x' there is an element $\vartheta \in Aut(G)$ such that $\vartheta(x) = x'$ if and only if there is an isomorphism $\varphi : G \to \overline{G}$ such that $\varphi(x) = x'$. It follows that each orbit of $Aut(G)$ is preserved under all isomorphisms from G onto $\overline{G}$. Thus, if A is a union of orbits, $G_{\restriction A} \in \mathcal{G}$. Since members of $\mathcal{G}$ have odd order, there is just one orbit, proving that $Aut(G)$ is vertex-transitive.

Let P be a self-complementary vertex-transitive graph. Clearly P is not of order 2. Let $x \in V(P)$. Since P is self-complementary, P_{-x} is isomorphic to $\overline{P}_{-y}$ for some $y \in V(P)$.

Since $Aut(\overline{P}) = Aut(P)$ and $Aut(P)$ is vertex-transitive, $\overline{P}_{-y}$ is isomorphic to $\overline{P}_{-x}$. Hence, $P \in \mathcal{G}$.

2. Let us recall that a *Paley graph* is a graph P_p whose vertices are the elements of $GF(p)$, the field of p elements, with $p \equiv 1$ $(mod\ 4)$; the edges are all pairs $\{x, y\}$ such that $x - y$ is a square in $GF(p)$. As it is well-known [15] (page 176), the automorphism group of P_p acts transitively on the edges and P_p is isomorphic to its complement. Hence, $P_p \in \mathcal{G}$.

3. Let $G, H \in \mathcal{G}$. Their *lexicographic product* is the graph $G.H$ obtained by replacing each vertex of H by a copy of G. Formally $V(G.H) := V(G) \times V(H)$ and $E(G.H)$ is the set of pairs $\{(u, v), (u', v')\}$ such that -either $v = v'$ and $\{u, u'\} \in E(G)$, -or $\{v, v'\} \in E(H)$. Let $x := (u, v) \in V(G.H)$. Select an isomorphism φ from G onto $\overline{G}$ which fixes u and an isomorphism ϑ from H onto $\overline{H}$ which fixes v. Set $\psi(u', v') := (\varphi(u'), \vartheta(v'))$. This defines an isomorphism from $G.H$ onto $\overline{G.H}$ which fixes x. Hence $G.H \in \mathcal{G}$ □

Questions 4.7 *Does $\mathcal{G}$ include a graph of order n whenever $n \equiv 1$ $(mod\ 4)$? In particular, is there such a graph of order 21?*

By Theorem 2.6 we have:

Remark 4.8 *Let G and G' be two graphs on the same set V of v vertices. Let k be an integer, $1 \leq k \leq v - 2$, $k \equiv 0$ $(mod\ 4)$. If G and G' are $(v - 1)$-hypomorphic and $e(G_{\restriction K})$ and $e(G'_{\restriction K})$ have the same parity up to complementation for all k-element subsets K of V, then either $G = G'$ or $G \in \mathcal{G}$.*

5 Conclusion

Let $\mathcal{R}$ be the set of pairs (v, k) such that two graphs on the same set are isomorphic up to complementation whenever these two graphs are k-hypomorphic up to complementation.

Behind Ille's problem was the question of a description of $\mathcal{R}$.

This seems to be a very difficult problem. Except the trivial inclusion $\mathcal{S} \subseteq \mathcal{R}$, the fact that some pairs like $(5, 4)$, $(v, v - 3)$ for $v \geq 7$ requires some effort [4].

We prefer to point out the following problem.

Problem 5.1 *Let $v > 2$. Is $(v, k) \in \mathcal{S} \iff 4 \leq k \leq \vartheta(v)$?*

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